

APPROXIMATI OF SOLUTIONS OF 2D NON-LINEAR INTEGRAL EQUATIONS IN A BINARY HILBERT SPACE

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Abstract

In this work, we investigate stability of solutions relative to the driver to a general class of nonlinear integral equations in two dimensions in binary reproducing kernel Hilbert space. Next, we analyze estimate error of solutions in terms of uniform grid numbers of compact subintervals of $[a, b] \times [c, d]$.

In addition, we use the reproducing kernel method to approximate the solutions of the problem. Finally, we provide an example to illustration the reliability of the method.

Keywords: Non-linear integral equation, Hilbert space, Reproducing kernel method.

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Introduction

Non-linear problems have Numerous applications in various branches of science, including physics, engineering and applied mathematics. Two-dimensional integral equations of the second kind can be used to model a variety of physics and engineering issues. [1–6]. We consider a general class of nonlinear Volterra integral equations in two dimensions:

$$u(x, y) = A(x, y, u(x, y)) + \iint_{ac}^{xy} F(x, y, s, t, u(s, t)) dt ds, \quad (1,1)$$

for all $(x, y) \in D = [a, b] \times [c, d]$; where " $u \in H(D)$ " is an unknown function to be determined, F satisfies the given regularity criterion and A is a given function in reproducing kernel Hilbert space $H(D)$; see Definition (2.6). Recently,

reproducing kernel Hilbert space method (RKHS) has been extensively studied and successfully applied to many real world problems such as in differential and integral equations, fractional differential equations, physics including fractional gas dynamics, advection–diffusion equation, fractional diffusion and advection–dispersion equations [2, 5–9]. Moreover, A mesh-free design and reproducing kernels that are piece-wise polynomial functions across compact intervals in $[a, b]$ and compact rectangles in $D = [a, b] \times [c, d]$ are two of RKHS's main advantages. A useful tool for numerical approaches is this.

The structure of this paper is as follows. In section 2, we introduce some fundamental terms, basic definitions, and notations related to the replicating kernel Hilbert spaces. Section 3 is devoted to the approximate solutions to the equation $\Lambda u = A$ where $\Lambda : H(D) \rightarrow H(D)$ is a linear transformation that is one-to-one and bounded. Section 4 investigates the local stability, estimate error, and uniform approximation of the $H(D)$ solutions to the suggested problem.

In section 5, we provide an example to demonstrate the process. Finally, we introduce the conclusions in section 6.

2. PRINCIPLES DEFINED

Recalling certain definitions and significant notations of replicating kernel Hilbert spaces is where we start. Regarding the information in this part, consult the references [7,8,10,11].

Definition 2.1 [11]: Let $X \neq \emptyset$, we say that H is a reproducing Kernel Hilbert space of continuous functions $h : X \rightarrow \mathbb{R}$ if there is a Bounded linear functional $L_x : H \rightarrow \mathbb{R}$, defined by $L_x(f) = f(x)$

$x \in X$.

The Riesz Representation Theorem guarantees that there exists a unique function $r : X \times X \rightarrow \mathbb{R}$ such that:

$$h(x) = L_x(h) = \langle h(\cdot), r(x, \cdot) \rangle_H \quad (2.1)$$

all $h \in H$ and all $x \in X$; $r = r(\cdot, \cdot)$ is alleged to be recreating H 's kernel function.

Definition 2.2. Let $W[a, b]$ be the space of all absolutely continuous functions " $g : [a, b] \rightarrow \mathbb{R}$ " such that g is square integrable on $[a, b]$.

Lemma 2.3 [7]: The space $W[a, b]$, equipped with the inner product

$$\langle f, g \rangle_{W[a,b]} = f(a)g(a) + \int_a^b f'(x)g'(x)dx,$$

Is a RKHS.

Remark 2.4 [10]: The reproducing kernel function $r = r(\cdot, \cdot)$ of $W[a, b]$ is given by

$$r_{(\xi)}(\tau) = \begin{cases} \xi + 1 - a & \text{if } a \leq \xi < \tau \leq b, \\ \tau + 1 - a & \text{if } a \leq \tau < \xi \leq b. \end{cases} \quad (2.2)$$

Definition 2.5 [8]: Let $f = f(\cdot, \cdot)$ be a function defined on D , and let

$I_k = \{[a_k, b_k] \times [c_k, d_k]\}_{k=1}^n$ be a finite set of non-overlapping domain

$I_k \subset D$. If for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{k=1}^n |f(I_k)| < \epsilon \quad \text{for} \quad \sum_{k=1}^n |I_k| < \delta,$$

Where $|f(I_k)| = f(b_k, d_k) - f(b_k, c_k) + f(a_k, c_k) - f(a_k, d_k)$ and $f(\cdot, c)$, $f(a, \cdot)$ are absolutely continuous functions with respect to x and y , respectively.

Then f is referred to as a totally continuous function on domain D .

Definition 2.6 [8]: The functions f mapping from D into $\mathbb{R}$ for which f is entirely continuous on D and the following conditions are true make up the space $H(D)$.

- 1) $f_x(\cdot, c)$ is square integrable on $[a, b]$;
- 2) $f_y(a, \cdot)$ is square integrable on $[c, d]$;
- 3) f_{xy} is square integrable on D .

Theorem 2.7 [7]: Let $\lambda \in \mathbb{R}$, $\alpha \in L^2[a, b]$, $\beta \in L^2$, $\gamma \in L^2(D)$,

$$u(x, y) = \lambda + \int_a^x \alpha(s) ds + \int_c^y \beta(t) dt + \iint_{ac}^{xy} \gamma(s, t) dt ds.$$

Then u belongs to $H(D)$.

Definition 2.8 [7]: The $H(D)$ space with the internal product

$$\begin{aligned} \langle g, h \rangle_{H(D)} &= g(a, c)h(a, c) + \int_a^b g_x(s, c)h_x(s, c) ds \\ &+ \int_c^d g_y(a, t)h_y(a, t) dt + \iint_{ac}^{bd} g_{xy}(s, t)h_{xy}(s, t) dt ds \end{aligned}$$

and align with convention

$$\|g\|_{H(D)} = \sqrt{\langle g, h \rangle_{H(D)}}$$

Lemma 2.9 [10]: The reproducing kernel function $k = k_{(\cdot, \cdot)}(\cdot, \cdot)$ of

the function space $H(D)$ is given by

$$k_{(s, t)}(x, y) = r_s(x)r_t(y), \quad (2.3)$$

Where $r = r(\cdot, \cdot)$ is defined by eq. (2.1).

3. Solution approximation.

In this section, for more information and proofs of the conclusions in this section, see [7, 8, 11]. We examine and present approximations of solutions to problem (1.1) in $H(D)$. Provide the definition of an integral operator from $H(D)$ into $H(D)$:

$$\Lambda u = u - Tu$$

where

$$Tu(x, y) = \iint_{ac}^{xy} F(x, y, s, t, u(s, t)) dt ds,$$

that $\Lambda u = A$.

Let $\{(s_i, t_i)\}_{i=1}^{\infty}$ be a dense set of points in D that can be counted, and let Λ^* stand for adjoint operator of Λ .

Remark 3.1. We define $\omega_i = \Lambda^* k_{(s_i, t_i)}$, where $k_{(s_i, t_i)}$ is the $H(D)$ kernel function and is described in eq. (2.3).

Theorem 3.2. If $\{(s_i, t_i)\}_{i=1}^{\infty}$ is a dense set of points on D . Then $\{\omega_i\}_{i=1}^{\infty}$ is a complete system of space $H(D)$ and $\omega_i = \Lambda k_{(s_i, t_i)}$ for all $i \in \mathbb{N}$

Proof. Let $\omega \in H(D)$, and let assume that $\langle \omega, \omega_i \rangle_{H(D)} = 0$ for all $i \in \mathbb{N}$. We see that

$$0 = \langle \omega, \omega_i \rangle_{H(D)} = \langle \omega, \Lambda^* k_{(s_i, t_i)} \rangle_{H(D)} = \langle \Lambda \omega, k_{(s_i, t_i)} \rangle_{H(D)} = \Lambda \omega(s_i, t_i),$$

for all $i \in \mathbb{N}$. Since $\Lambda \omega \in H(D)$; $H(D)$ is the space of continuous functions on D . As a result, $\Lambda \omega$ vanishes at a dense set of points in D , this implies that ω is equivalent to zero. Therefore, $\{\omega_i\}_{i=1}^{\infty}$ is a complete system in $H(D)$.

Next assertion, it follows from the symmetry properties of the reproducing kernel function k that

$$k_{(s, t)}(p, q) = k_{(p, q)}(s, t),$$

For all $p, q, s, t \in D$. Next,

$$\begin{aligned} \Lambda^* k_{(s, t)}(p, q) &= \langle \Lambda^* k_{(s, t)}(\cdot, \cdot), k_{(p, q)}(\cdot, \cdot) \rangle_{H(D)} \\ &= \langle k_{(s, t)}(\cdot, \cdot), \Lambda k_{(p, q)}(\cdot, \cdot) \rangle_{H(D)} \\ &= \langle \Lambda k_{(p, q)}(\cdot, \cdot), k_{(s, t)}(\cdot, \cdot) \rangle_{H(D)} \\ &= \Lambda k_{(p, q)}(s, t) \\ &= \Lambda k_{(s, t)}(p, q). \end{aligned}$$

If we choose $(s, t) = (s_i, t_i)$ for all $i \in \mathbb{N}$.

Then it follows from equation (3.1) that

$$\omega_i = \Lambda^* k_{(s_i, t_i)} = \Lambda k_{(s_i, t_i)}.$$

Clearly, an orthogonal system $\{Y_i\}_{i=1}^{\infty}$ can be derived by using the Gram-Schmidt orthogonalization process to $\{\omega_i\}_{i=1}^{\infty}$ as:

$$Y_j = \sum_{l=1}^n c_{jl} \omega_l,$$

where the c_{jl} are orthonormalization coefficients of $\{\omega_i\}_{i=1}^{\infty}$.

Theorem 3.3. Let $\{(s_i, t_i)\}_{i=1}^{\infty}$ be a dense set of points of D , and let $u \in H(D)$ be a solution of problem (1.1). Then u has the following Hilbert space description:

$$u = \sum_{j=1}^{\infty} \sum_{l=1}^j c_{jl} A(s_l, t_l, u_l) Y_j$$

Proof. Since $\{Y_i\}_{i=1}^{\infty}$ is a complete orthonormal basis in $H(D)$. One can write

$$u = \sum_{j=1}^{\infty} \langle u, Y_j \rangle_{\mathbb{H}} Y_j.$$

Then, it follows from equation (3.2) that

$$u = \sum_{j=1}^{\infty} \sum_{l=1}^j c_{jl} \langle u, \omega \rangle_{\mathbb{H}} Y_j$$

we conclude for all $l \in \mathbb{N}$ that

$$\begin{aligned} \langle u, \omega_l \rangle_{\mathbb{H}} &= \langle u, \Lambda^* k_{(s_l, t_l)} \rangle_{\mathbb{H}} = \langle \Lambda u, k_{(s_l, t_l)} \rangle_{\mathbb{H}} \\ &= \langle A, k_{(s_l, t_l)} \rangle_{\mathbb{H}} = A(s_l, t_l, u_l). \end{aligned}$$

Therefore, we obtain the desired result

$$u = \sum_{j=1}^{\infty} \sum_{l=1}^j c_{jl} A(s_l, t_l, u_l) Y_j.$$

Remark 3.4. By replacing the infinite sum in the representation of solution (3.3) by a finite one, so we obtain the approximate solution u_n to the problem (1.1) as follows:

$$u_n = \sum_{j=1}^n \sum_{l=1}^j c_{jl} A(s_l, t_l, u_l) Y_j$$

4. Solution Stability and Error Estimation

In this section in $H(D)$ with respect to the driver A in H , we investigate the global stability for the solution to issue (1.1). (D). Additionally, we analyze the bounded uniform error when the exact solution u in $H(D)$ to issue is approximated by the truncation solution u_n (1.1). We observe that the operator notation for problem (1.1) is defined in previous section:

$$\Lambda u = u - Tu = A.$$

Theorem 4.1. Let A_1, A_2 belong to $H(D)$, and let u_1, u_2 be the unique solutions in $H(D)$ to $\Lambda u_j = A_j$ for all $(j = 1, 2)$. Then there corresponds a constant $\lambda > 0$ with the property that:

$$\|u_1 - u_2\|_{H(D)} \leq \lambda \|A_1 - A_2\|_{H(D)}.$$

Proof. Since,

$$\begin{aligned} \|u_1 - u_2\|_{H(D)} &= \|A_1 + Tu_1 - (A_2 + Tu_2)\|_{H(D)} \\ &= \|A_1 - A_2 + Tu_1 - Tu_2\|_{H(D)} \\ &\leq \|A_1 - A_2\|_{H(D)} + \|Tu_1 - Tu_2\|_{H(D)} \end{aligned}$$

It follows that

$$\|u_1 - u_2\|_{H(D)} \leq \|A_1 - A_2\|_{H(D)} + \delta_{\Delta} \|u_1 - u_2\|_{H(D)}$$

Implies that,

$$\|u_1 - u_2\|_{H(D)} \leq \frac{1}{1 - \delta_{\Delta}} \|A_1 - A_2\|_{H(D)},$$

Hence,

$$\|u_1 - u_2\|_{H(D)} \leq \lambda \|A_1 - A_2\|_{H(D)},$$

where $\lambda \leq \frac{1}{1 - \delta_{\Delta}}$ and $\delta_{\Delta} \leq C\sqrt{\Delta x \Delta y}$; C is a positive constant. Obviously,

because Δx and Δy are sufficiently small or can be taken small enough

such that $\delta\Delta < 1$.

To determine the error estimate result and upper bound for the error on D for solutions to the problem (1.1). Before starting the next theorem, we need to define an integral operator $\Pi: \mathbb{H}(D) \rightarrow \mathbb{H}(D)$ as

$$\Pi u = A + Tu, \quad (4.1)$$

this implies that $\Pi u = u$ for all $u \in \mathbb{H}(D)$.

Theorem 4.2. Let $n > 0$, $x_i, y_i \in D$, "where" $x_i = x_{i-1} + \Delta x$ "and

" $y_i = y_{i-1} + \Delta y$ " for all $(i = 1, 2, \dots, n)$ with " $\Delta x = \frac{b-a}{n}$ " and " $\Delta y = \frac{c-d}{n}$."

Let u be a unique solution in $\mathbb{H}(D)$ to $\Pi u = u$, and let u_n be an approximate solution of u given by remark (3.4). Then

$$|u(x, y) - u_n(x, y)| \leq \lambda \|A\|_{\mathbb{H}(D)},$$

for some positive constant λ and for all $x, y \in D$.

Proof. For each $x, y \in D$, $n > 0$ then there exist " $x_i, y_i \in D_i$," such that " $|x - x_i| < \frac{1}{n}$ and $|y - y_i| < \frac{1}{n}$." Applying

$$\Pi u(x_i, y_i) = \Pi u_n(x_i, y_i) \text{ for } (i = 1, 2, \dots, N)$$

implies that

$$\begin{aligned} |u(x, y) - u_n(x, y)| &= |u(x, y) - \Pi u(x_i, y_i) + \Pi u_n(x_i, y_i) - u_n(x, y)| \\ &= |u(x, y) - u(x_i, y_i) + u_n(x_i, y_i) - u_n(x, y)| \\ &\leq |u(x, y) - u(x_i, y_i)| + |u_n(x_i, y_i) - u_n(x, y)|. \end{aligned}$$

By definition of reproducing kernel Hilbert space (2.1) we get

$$\begin{aligned} |u(x, y) - u(x_i, y_i)| &= |\langle u(\cdot, \cdot), k_{(x, y)}(\cdot, \cdot) \rangle_{\mathbb{H}} - \langle u(\cdot, \cdot), k_{(x_i, y_i)}(\cdot, \cdot) \rangle_{\mathbb{H}}| \\ &\leq |\langle u(\cdot, \cdot), k_{(x, y)}(\cdot, \cdot) - k_{(x_i, y_i)}(\cdot, \cdot) \rangle_{\mathbb{H}}| \\ &\leq \|\Pi u\|_{\mathbb{H}} \|k_{(x, y)}(\cdot, \cdot) - k_{(x_i, y_i)}(\cdot, \cdot)\|_{\mathbb{H}} \end{aligned}$$

Since $k_{(s, t)}(x, y) = r_s(x)r_t(y)$ is defined by (2.3) and using properties of an inner product we get:

$$\begin{aligned} |u(x, y) - u(x_i, y_i)| &\leq \|\Pi u\|_{\mathbb{H}} \|r_x(\cdot)r_y(\cdot) - r_{x_i}(\cdot)r_{y_i}(\cdot)\|_{\mathbb{H}} \\ &= \|\Pi u\|_{\mathbb{H}} \|r_x(\cdot)r_y(\cdot) + r_{x_i}(\cdot)r_y(\cdot) - r_{x_i}(\cdot)r_y(\cdot) - r_{x_i}(\cdot)r_{y_i}(\cdot)\|_{\mathbb{H}} \\ &\leq \|\Pi u\|_{\mathbb{H}} \|r_y(\cdot)\|_{\mathcal{W}} \|r_x(\cdot) - r_{x_i}(\cdot)\|_{\mathcal{W}} \\ &\quad + \|\Pi u\|_{\mathbb{H}} \|r_{x_i}(\cdot)\|_{\mathcal{W}} \|r_y(\cdot) - r_{y_i}(\cdot)\|_{\mathcal{W}} \end{aligned}$$

It follows from Lemma 2.3 (page 2) in [7] that

$$\begin{aligned} |u(x, y) - u(x_i, y_i)| &\leq C \|\Pi u\|_{\mathbb{H}(D)} (\sqrt[2]{|x - x_i|} + \sqrt[2]{|y - y_i|}) \\ &\leq C \|\Pi u\|_{\mathbb{H}(D)} (\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}}) \\ &= \frac{2C}{\sqrt{n}} \|\Pi u\|_{\mathbb{H}(D)} \end{aligned} \quad (4.2)$$

Where $C = \max \{\|r_{x_i}(\cdot)\|, \|r_y(\cdot)\|\}$. Similarly, one can use the same technique to get

$$|u_n(x, y) - u_n(x_i, y_i)| \leq \frac{2C}{\sqrt{n}} \|\Pi u\|_{\mathbb{H}(D)} \quad (4.3)$$

Combine eq. (4.2) and (4.3), we get the final result

$$|u(x, y) - u_n(x, y)| \leq \lambda \| \Pi u \|_{\mathbb{H}(D)},$$

Where $\lambda \leq \frac{4C}{\sqrt{n}}$ and For all $x, y \in D$.

In this section, we solve numerically two dimensional non-linear Volterra integral equation in the space $H(D)$ by applying reproducing kernel method. The results indicate in table 1 that shows the reproducing kernel method is efficient.

5. Numerical experiments

Example 5.1. Consider two dimensional non-linear Volterra integral equation

$$u(x, y) = A(x, y, u) + \iint_{00}^{xy} (xys, t, u(s, t))^2 dt ds,$$

where $A(x, y, u) = u(x, y) \cdot \frac{yx^5}{6} (2y^2 + 3e^y - 3) - \frac{x^5 e^y}{5} \sinh(y)$ and we note that the exact solution for this problem is $u(x, y) = x^2 e^y$.

Taking $x_i = \frac{i}{n}$ and $y_i = \frac{i}{n}$ for all $(i=0, 1, 2, \dots, 10)$ with the reproducing

kernel function $k_{(s,t)}(x, y) = r_s(x) r_t(y)$ on $[0, 1] \times [0, 1]$. The approximate solution $u_N = u_N(\cdot, \cdot)$ is evaluated by (3.4). The numerical results are showed in Table 1 and 2 with selected $n = 8, 10$, and $n = 15$ uniformly grid points in $[0, 1] \times [0, 1]$. We see from Theorem 4.2 that the upper bound for the uniform error in $[0, 1] \times [0, 1]$ are given as:

$$|u(x, y) - u_8(x, y)| \leq \frac{2 \| \Pi u \|_{\mathbb{H}(D)}}{\sqrt{8}} \approx 0.01;$$

$$|u(x, y) - u_{10}(x, y)| \leq \frac{2 \| \Pi u \|_{\mathbb{H}(D)}}{\sqrt{10}} \approx 0.001;$$

$$|u(x, y) - u_{15}(x, y)| \leq \frac{2 \| \Pi u \|_{\mathbb{H}(D)}}{\sqrt{15}} \approx 0.00001$$

Table 5.1: Numerical results and errors estimation

(x,y)	n=8	n=10	n=15
(0.0,0.0)	0	0	0
(0.1,0.1)	0.17×10^{-3}	0.17×10^{-4}	2.4×10^{-6}
(0.2,0.2)	0.29×10^{-3} 10^{-4}	$0.29 \times$ 10^{-6}	$2.03 \times$
(0.3,0.3)	0.40×10^{-3} 10^{-5}	$0.40 \times$ 10^{-7}	$3.01 \times$
(0.4,0.4)	0.50×10^{-3} 10^{-5}	$0.50 \times$ 10^{-7}	$1.50 \times$
(0.5,0.5)	0.59×10^{-3}	0.59×10^{-6} 10^{-7}	$9.10 \times$
(0.6,0.6)	0.68×10^{-3}	0.68×10^{-6} 10^{-8}	$1.10 \times$
(0.7,0.7)	0.76×10^{-3}	0.76×10^{-6} 10^{-7}	$9.10 \times$
(0.8,0.8)	0.84×10^{-3}	0.84×10^{-5} 10^{-7}	$2.40 \times$
(0.9,0.9)	0.92×10^{-3}	0.92×10^{-5} 10^{-6}	$1.53 \times$
(1.0,1.0)	0.23×10^{-3}	0.23×10^{-2} 10^{-5}	$5.10 \times$
(0.0,0.0)	0	0	0
(0.1,0.1)	0.17×10^{-3}	0.17×10^{-4}	2.4×10^{-6}
(0.2,0.2)	0.29×10^{-3} 10^{-4}	$0.29 \times$ 10^{-6}	$2.03 \times$
(0.3,0.3)	0.40×10^{-3} 10^{-5}	$0.40 \times$ 10^{-7}	$3.01 \times$
(0.4,0.4)	0.50×10^{-3} 10^{-5}	$0.50 \times$ 10^{-7}	$1.50 \times$
(0.5,0.5)	0.59×10^{-3}	0.59×10^{-6} 10^{-7}	$9.10 \times$
(0.6,0.6)	0.68×10^{-3}	0.68×10^{-6} 10^{-8}	$1.10 \times$
(0.7,0.7)	0.76×10^{-3}	0.76×10^{-6} 10^{-7}	$9.10 \times$
(0.8,0.8)	0.84×10^{-3}	0.84×10^{-5} 10^{-7}	$2.40 \times$
(0.9,0.9)	0.92×10^{-3}	0.92×10^{-5} 10^{-6}	$1.53 \times$
(1.0,1.0)	0.23×10^{-3}	0.23×10^{-2} 10^{-5}	$5.10 \times$

Table5. 2: Numerical results and Relative errors

(x ,y)	n=8	n=10	n=15
(0.0,0.0)	0	0	0
(0.1,0.1)	0.07×10^{-3}	0.01×10^{-4}	0.4×10^{-6}
(0.2,0.2)	0.19×10^{-4}	0.22×10^{-5}	10^{-6}
(0.3,0.3)	0.10×10^{-4}	0.20×10^{-6}	10^{-8}
(0.4,0.4)	0.01×10^{-4}	0.1×10^{-6}	10^{-8}
(0.5,0.5)	0.2×10^{-4}	0.11×10^{-6}	10^{-7}
(0.6,0.6)	0.34×10^{-6}	0.24×10^{-7}	10^{-9}
(0.7,0.7)	0.25×10^{-4}	0.44×10^{-6}	10^{-7}
(0.8,0.8)	0.34×10^{-4}	0.34×10^{-6}	10^{-8}
(0.9,0.9)	0.28×10^{-4}	0.12×10^{-6}	10^{-7}
(1.0,1.0)	0.11×10^{-2}	0.37×10^{-4}	10^{-5}

6. Conclusion

We discussed stability of solutions with respect to the driver A to problem (1.1). Furthermore, we studied error estimate to the problem interms of uniform node points on compact subintervals of D and we

showed that the upper bounded error is not exceed $\lambda \|A\|_{\mathbb{H}(D)}$.

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